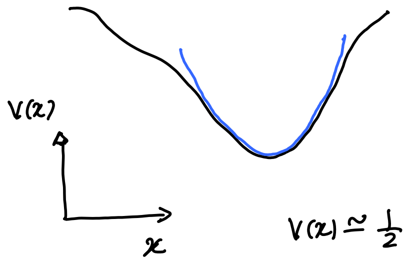
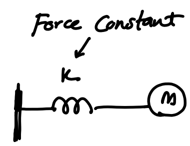
Harmonic Oscillator (HO)

Note Title 3/4/2008





$$V(x) \simeq \frac{1}{2} k x^2$$

$$H = T + V$$

$$= \frac{p^2}{2m} + \frac{1}{2} k x^2 \qquad \omega = \sqrt{\frac{k}{m}} \rightarrow k = m\omega^2$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = \frac{m\omega^2}{2}\left(\frac{\hat{p}^2}{m^2\omega^2} + \hat{x}^2\right)$$

Check the identity:

$$(a+ib)(a-ib) + (a-ib)(a+ib) =$$

$$\hat{a}^2 + \hat{b}^2 + i\cancel{\hat{b}\hat{a}} - i\cancel{\hat{a}\hat{b}} + \hat{a}^2 + \hat{b}^2 + \cancel{i\hat{a}\hat{b}} - \cancel{i\hat{b}\hat{a}} = 2(\hat{a}^2 + \hat{b}^2)$$

Use this relation:

$$\hat{H} = \frac{1}{2}\frac{m\omega^2}{2}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) + \frac{1}{2}\frac{m\omega^2}{2}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

Define:

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad \hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

$$\hookrightarrow \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{+}\right)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^{+} - \hat{a}\right)$$

$$\Rightarrow \boxed{\hat{H} = \frac{1}{2}\left(\hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a}\right)}$$

$\hat{a}$ & $\hat{a}^{+}$ are related to $\hat{x}$ & $\hat{p}$ as above.

Let's check some properties of $\hat{a}$ & $\hat{a}^{+}$:

$$[\hat{a}, \hat{a}^{+}] = ?$$

$$[\hat a, \hat a^+] = \hat a \hat a^+ - \hat a^+ \hat a = ?$$

$$[\hat a, \hat a^+]\psi = \hat a \hat a^+ \psi - \hat a^+ \hat a \psi \qquad \hat p = -i\hbar\frac{\partial}{\partial x}$$

$$= \frac{m\omega}{2\hbar}\left(\hat x + \frac{i\hat p}{m\omega}\right)\left(\hat x - \frac{i\hat p}{m\omega}\right)\psi - \frac{m\omega^2}{2}\left(\hat x - \frac{i\hat p}{m\omega}\right)\left(\hat x + \frac{i\hat p}{m\omega}\right)\psi$$

$$= \frac{m\omega}{2\hbar}\left\{\underbrace{\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi}_{I} - \underbrace{\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)x\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)}_{II}\right\}$$

$$I = x^2\psi - \frac{\hbar}{m\omega}x\frac{\partial}{\partial x}\psi + \underline{\frac{\hbar}{m\omega}\frac{\partial}{\partial x}}x\psi - \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\psi$$

$$= x^2\psi - \cancel{\frac{\hbar}{m\omega}x\psi'} + \underline{\frac{\hbar}{m\omega}\psi + \cancel{\frac{\hbar}{m\omega}x\psi'}} - \frac{\hbar^2}{m\omega^2}\psi''$$

$$II = -x^2\psi - \frac{\hbar}{m\omega}x\frac{\partial}{\partial x}\psi + \underline{\frac{\hbar}{m\omega}\frac{\partial}{\partial x}}x\psi + \frac{\hbar^2}{m^2\omega^2}\frac{\partial^2}{\partial x^2}\psi$$

$$= -x^2\psi - \cancel{\frac{\hbar}{m\omega}x\psi'} + \underline{\frac{\hbar}{m\omega}\psi + \cancel{\frac{\hbar}{m\omega}x\psi'}} + \frac{\hbar^2}{m\omega^2}\psi''$$

Substitute in $[\hat a, \hat a^+] \Rightarrow$

$$[\hat a, \hat a^+] = \frac{m\omega}{2\hbar}[I + II] = \frac{m\omega}{2\hbar}\,\frac{2\hbar}{m\omega} = 1$$

$$\boxed{[\hat a, \hat a^+] = 1} \quad \Rightarrow \quad \hat a \hat a^+ = 1 + \hat a^+ \hat a$$

$$[\hat a^+, \hat a] = -1$$

$$[\hat a, \hat a] = 0$$

$$[\hat a^+, \hat a^+] = 0$$

$$\hat H = \frac{\hbar\omega}{2}(\hat a \hat a^+ + \hat a^+ \hat a) = \frac{\hbar\omega}{2}(1 + a^+a + a^+a)$$

$$\boxed{\hat H = \hbar\omega\left(a^+a + \frac{1}{2}\right)}$$

Let's now solve the schrodinger equation & find the ground state :

$$\hat{H}\psi_n = E_n\psi_n$$

$$\hat{a} \times \left(\hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})\psi_n = E_n\psi_n \right)$$

$$\hbar\omega(\hat{a}\hat{a}^+\hat{a} + \tfrac{1}{2}\hat{a})\psi_n = E_n\hat{a}\psi_n$$

$$\hbar\omega(\underbrace{\hat{a}\hat{a}^+}_{1+a^+a} + \tfrac{1}{2})(\hat{a}\psi_n) = E_n(\hat{a}\psi_n)$$

$$\hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})\hat{a}\psi_n + \hbar\omega\hat{a}\psi_n = E_n\hat{a}\psi_n$$

$$\underbrace{\hbar\omega(\hat{a}^+\hat{a} + \tfrac{1}{2})}_{\hat{H}}\underbrace{\hat{a}\psi_n}_{\psi_{n-1}} = \underbrace{(E_n - \hbar\omega)}_{E_{n-1}}\underbrace{\hat{a}\psi_n}_{\psi_{n-1}}$$

we have defined $\psi_{n-1} \equiv \hat{a}\psi_n$. This is also an eigenfunction of the Hamiltonian $\hat{H} = \hat{a}^+\hat{a} + \frac{1}{2}$ with eigenvalue of $E_{n-1} = E_n - \hbar\omega$.

So we have shown that operator $\hat{a}$ by acting on schrodinger equation, creates a new eigenfunction with eigenenergy $E_n - \hbar\omega$.

Similarly we can show that $\hat{a}^+$ by acting on schrodinger equation, creates a new eigenfunction with eigenenergy $E_n + \hbar\omega$